

Unified description of cosmological and static solutions in affine generalized theories of gravity: vecton - scalaron duality and its applications.

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April 15, 2013

Abstract

We briefly describe the simplest class of affine theories of gravity in multidimensional space-times with symmetric connections and their reductions to two-dimensional *dilaton - vecton gravity* field theories (DVG). The distinctive feature of these theories is the presence of an absolutely neutral massive (or tachyonic) vector field (*vecton*) with essentially nonlinear coupling to the dilaton gravity (DG). We show that in DVG the vecton field can be consistently replaced by an effectively massive scalar field (*scalaron*) with an unusual coupling to dilaton gravity. With this vecton - scalaron duality, one can use methods and results of the standard DG coupled to usual scalars (DGS) in more complex *dilaton - scalaron gravity* theories (DSG) equivalent to DVG. We present the DVG models derived by reductions of multidimensional affine theories and obtain one-dimensional dynamical systems simultaneously describing cosmological and static states in any gauge. Our approach is fully applicable to studying static and cosmological solutions in multidimensional theories as well as in general one-dimensional DGS models. We focus on global properties of the models, look for integrals and analyze the structure of the solution spaces. In integrable cases, it can be usefully visualized by drawing a ‘topological portrait’ resembling phase portraits of dynamical systems and simply exposing global properties of static and cosmological solutions, including horizons, singularities, etc. For analytic approximations we also propose an integral equation well suited for iterations.

1 Introduction

The present observational data strongly suggest that Einstein’s gravity must be modified. The combination of data on dark energy (DE) and growing evidence for some sort of inflation have generated a wide spectrum of such modifications (see, e.g., [1] - [10]). Superstring ideas suggested natural modifications but, in view of the serious mathematical problems of the present string theory, strictly definite predictions about concrete modifications of gravity are not yet available. Moreover, the phenomenon of dark energy was not predicted by string theory and its origin in the stringy framework proved to be rather difficult to uncover and understand. The problem of dark energy in string theory looks very deep and is related to many other complex issues of (non-existing) quantum cosmology but it also led to some beautiful and exciting speculations, like eternal inflation and multiverse (see, e.g., [11] and [12]). On the other hand, if we modestly try first to find a natural place for DE in classical cosmological models, which inevitably are essentially nonlinear and non-integrable, we better return to recalling the origin of general relativity and look for some options abandoned or not found by its creators.

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For this reason, simpler modifications of gravity that affect only the gravitational sector are also popular, e.g., [13] - [16]. In essence, these modifications reduce to the standard Einstein gravity supplemented by some number of scalar bosons (the first example of such a modification was the old Jordan - Brans - Dicke theory). The main problem of this approach is that the origin of these scalar bosons is quite obscure, and there is no theoretical principle regulating their coupling to gravity. Of course, there exist some phenomenological and theoretical constraints but in general the spectrum of these models is too wide.¹ The modification proposed and studied in [18] - [22] satisfy some principles of geometric nature based on Einstein's idea (1923)² to formulate the gravity theory in a non-Riemannian space with a symmetric connection determined by a special variational principle involving a 'geometric' Lagrangian. This Lagrangian is assumed to be a function of the generalized Ricci curvature tensor as well as of other fundamental tensors and is varied in the connection coefficients. A new interpretation and generalization of this approach was developed in [19] - [21] for an arbitrary space-time dimension D .

The connection coefficients define symmetric and antisymmetric parts of the Ricci tensor (s_{ij} and a_{ij}) and a new vector a_i . Supposing that in the pure geometry there is no dimensional fundamental constants (except the velocity of light relating space to time) we choose geometric Lagrangians giving dimensionless geometric action. The geometric variational principle puts further bounds on the geometry and, in particular, relates a_i to a_{ij} . To define a metric tensor we have to introduce some dimensional constant. Then we can find a physical Lagrangian depending on this dimensional constant and on some dimensionless parameters. Thus obtained theory supplements the standard general relativity with dark energy (the cosmological term, in the limit $a_i = a_{ij} = 0$), neutral massive (or tachyonic) vector field proportional to a_i (vorton) and, after dimensional reductions to $D = 4$, with $(D - 4)$ massive (or tachyonic) scalar fields.

Further properties of the theory depend on a concrete choice of the *geometric Lagrangian*. The resulting physical theory can be described by the corresponding *effective Lagrangian* that depends on the metric, vorton and scalar fields. In the geometric theory, there are no dimensional constants, while any fundamental tensor has dimension of some power of length (assuming $c = 1$). We proposed a class of geometric Lagrangian densities depending on the symmetric and antisymmetric parts of the generalized Ricci tensor, s_{ij} , a_{ij} , and on a fundamental vector obtained by contracting the connection coefficients Γ_{ij}^k , i.e., $a_i \sim \Gamma_{ik}^k$. By requiring the geometric 'action' (the space-time integral of the geometric Lagrangian density) to be dimensionless we can enumerate all possible actions, see [19]. The corresponding Lagrangian may be defined as the square root of an arbitrary linear combination of these densities or a linear combination of their square roots.

The most natural density of this sort in any dimension is the square root of $\det(s_{ij} + \bar{l}a_{ij})$, where \bar{l} is a number.³ The effective physical Lagrangian is the sum of the standard Einstein term, the vorton mass term, and the term proportional to $\det(g_{ij} + lf_{ij})$ to the power $\nu \equiv 1/(D - 2)$, where g_{ij} and f_{ij} are the metric and the vorton field tensors *conjugate* to s_{ij} and a_{ij} ,⁴ l is a parameter related to \bar{l} . The last term has the dimensional multiplier, which in the limit of small field f_{ij} produces the cosmological constant. For $D = 4$ we therefore have the term first introduced by Einstein but now usually called the Born - Infeld or brane Lagrangian. For $D = 3$ we have the Einstein - Proca theory, which is very interesting for studies of nontrivial space topologies.

The simplest dimensional reductions from $D > 4$ to $D = 4$ produce $(D - 4)$ scalar fields and thus the complete theory is rather complex, even at the classical level. Its spherically symmetric sector is described by a much simpler (1+1)-dimensional dilaton gravity coupled to one massive vector and to several scalar fields. This dilaton gravity coupled to the vorton and massive scalars

¹Restricting consideration to homogenous cosmologies, one can find that in dimensionally reduced supergravity (superstring) theory there emerge many massless scalar bosons that couple to gravity only, see, e.g., [17], [2].

²References to his papers as well as to related work of Weyl and Eddington can be found in [18] - [21].

³Einstein used as the Lagrangian Eddington's scalar density $\sqrt{|\det r_{ij}|}$, where $r_{ij} \equiv s_{ij} + a_{ij}$.

⁴This unusual construction introduced by A. Einstein is described and generalized in [18] - [21], see also Appendix.

as well as its further reductions to one-dimensional ‘cosmological’ and ‘static’ theories were first formulated in [18] - [21], and here we begin systematic studies of their general properties and solutions. These studies can be somewhat simplified by transforming the vecton field into a new massive scalar field, which is possible (on the mass shell) in the two-dimensional reductions.⁵ As distinct from the normal scalar matter fields, the scalaron has a different coupling to gravity and may have abnormal signs of the kinetic term (phantom) or mass term (tachyon). Nevertheless, some general methods used in DGS models can be successfully applied to DG coupled to abnormal scalar fields. Thus, the main purposes of the present study are: to essentially modernize available tools for constructing various approximation schemes, to understand most important qualitative features of the global solutions, and to look for possible integrable approximation in simple one-dimensional reductions of the general theory.

In this paper, we consider the simplest *geometric Lagrangian*,

$$\mathcal{L}_{\text{geom}} = \sqrt{-\det(s_{ij} + \bar{l}a_{ij})} \equiv \sqrt{-\Delta_s}, \quad (1)$$

where the minus sign is taken because $\det(s_{ij}) < 0$ (due to the local Lorentz invariance) and we naturally assume that the same is true for $\det(s_{ij} + \bar{l}a_{ij})$ (to reproduce Einstein’s general relativity with the cosmological constant in the limit $\bar{l} \rightarrow 0$). Following the steps of Ref. [19] or using results described in Appendix 7.1 one can derive the corresponding *physical Lagrangian*

$$\mathcal{L}_{\text{phys}} = \sqrt{-g} \left[-2\Lambda [\det(\delta_i^j + lf_i^j)]^\nu + R(g) - m^2 g^{ij} a_i a_j \right], \quad \nu \equiv 1/(D-2), \quad (2)$$

which should be varied with respect to the metric and the vector field; m^2 is a parameter depending on the chosen model for affine geometry and on D (see [18] - [21]). This parameter can be positive or negative and we often use notation $m^2 \equiv \mu$. When the vecton field vanishes, we have the standard Einstein gravity with the cosmological constant. For dimensional reductions from $D \geq 5$ to $D = 4$, we can obtain the Lagrangian describing the vecton a_i , $f_{ij} \sim \partial_i a_j - \partial_j a_i$ and $(D-4)$ scalar fields a_k , $k = 4, \dots, D$. Note that, for $D = 3$, the Lagrangian (2) is bilinear in the vecton field, and in the approximation $a_i = 0$ it gives the three-dimensional gravity with the cosmological term. The three-dimensional gravity was studied by many authors (see, e.g., [23], [24] and references therein) and this may significantly simplify the study of solutions of the new theory.

We first consider the simplest spherical dimensional reduction from the D -dimensional theory (2) to the two-dimensional dilaton-vecton gravity (DVG) and then further reduce it to one-dimensional static or cosmological theories. The next step is considering cylindrical reductions of the D -dimensional theory to more complex two-dimensional dilaton gravity theories coupled to several scalar fields. For simplicity, we consider this dimensional reduction only in the dimensions four and three. As is well known, in the $D = 4$ case, in addition to the dilaton, there appear ‘geometric’ σ -model scalar fields and vector pure gauge fields that produce an effective potential introduced in [25]. These two-dimensional scalars look like scalar ‘matter’ fields obtained by dimensional reductions from higher dimensional supergravity theory. Like those fields, they are classically massless. However, in general, the σ -model coupling of the scalar fields is supplemented by the mentioned effective potential depending on the scalar fields and on the ‘charges’ of the ‘geometric’ pure gauge fields. This theory is rather complex and not integrable even after reduction to dimensions 1+0 or 0+1 (see [25]). Adding the vecton coupling makes the theory even more complex and in this paper we consider only the simplest variants.

⁵In addition to some formal simplifications, the transformation may shed a new light on some theoretical problems, especially, in cosmology. Indeed, new cosmological theories operate with several exotic scalar fields (inflavons, phantoms, tachyons), which are usually introduced *ad hoc*. In the models considered here, such ‘particles’ appear as effective fields in a consistent theoretical framework.

Below, we mainly discuss low-dimensional theories ($D = 1, 2, 3, 4$) that allow us to consistently treat some solutions of realistic higher dimensional theories. We mostly concentrate on mathematical problems and do not pay much attention to physics meaning of the obtained solutions. Note also that in the context of modern ideas on inflation, multiverse, etc., the main parameters of a fundamental theory of gravity cannot be theoretically determined. In particular, we do not know the sign and the magnitude of the cosmological constant and of other parameters giving a scale of length. Even the dimension and signature of the space-time should be considered as a free parameters that can be estimated in the context of a concrete scenario of the multiverse evolution or by anthropic considerations. In practice, this means that we should regard theories with any parameters in any space-time dimension as equally interesting, at least, theoretically.

2 Dimensional reductions of the generalized gravity theory

Let us outline the main reductions of the model (2) in the dimensions $D = 3, 4$.⁶ Due to natural space and time restrictions we only give an overview of main points. First, consider a rather general Lagrangian in the D -dimensional *spherically symmetric* case ($x^0 = t$, $x^1 = r$):

$$ds_D^2 = ds_2^2 + ds_{D-2}^2 = g_{ij} dx^i dx^j + \varphi^{2\nu} d\Omega_{D-2}^2, \quad (3)$$

where $\nu \equiv (D - 2)^{-1}$. The standard spherical reduction of (2) gives the effective Lagrangian⁷ the first three terms of which describe the standard spherically reduced Einstein gravity:

$$\mathcal{L}_D^{(2)} = \sqrt{-g} \left[\varphi R(g) + k_\nu \varphi^{1-2\nu} + W(\varphi) (\nabla\varphi)^2 + X(\varphi, \mathbf{f}^2) - m^2 \varphi \mathbf{a}^2 \right]. \quad (4)$$

Here $a_i(t, r)$ has only two non-vanishing components a_0, a_1 , f_{ij} has just one independent component $f_{01} = a_{0,1} - a_{1,0}$; the other notations are: $\mathbf{a}^2 \equiv a_i a^i \equiv g^{ij} a_i a_j$, $\mathbf{f}^2 \equiv f_{ij} f^{ij}$, $k_\nu \equiv k(D - 2)(D - 3)$, $W(\varphi) = (1 - \nu)/\varphi$ and, finally,⁸

$$X(\varphi, \mathbf{f}^2) \equiv -2\Lambda\varphi \left[1 + \frac{1}{2} \lambda^2 \mathbf{f}^2 \right]^\nu, \quad (5)$$

where, the parameter λ is related to dimensionless parameter l in (2) but $[\lambda] = \text{L}$.

Sometimes, it is convenient to transform away the dilaton kinetic term by using the Weyl transformation, which in our case is the following ($w'(\varphi)/w(\varphi) = W(\varphi)$):

$$g_{ij} = \hat{g}_{ij} w^{-1}(\varphi), \quad w(\varphi) = \varphi^{1-\nu}, \quad \mathbf{f}^2 = w^2 \hat{\mathbf{f}}^2, \quad \mathbf{a}^2 = w \hat{\mathbf{a}}^2. \quad (6)$$

Applying this transformation to (4) and omitting the hats we find the transformed Lagrangian

$$\hat{\mathcal{L}}_D^{(2)} = \sqrt{-g} \left[\varphi R(g) + k_\nu \varphi^{-\nu} - 2\Lambda\varphi^\nu \left(1 + \frac{1}{2} \lambda^2 \varphi^{2(1-\nu)} \mathbf{f}^2 \right)^\nu - m^2 \varphi \mathbf{a}^2 \right]. \quad (7)$$

When $D = 3$ we have $\nu = 1$, $k_\nu = 0$, Weyl's transformation is trivial and the Lagrangian is

$$\mathcal{L}_3^{(2)} = \sqrt{-g} \varphi \left[R(g) - 2\Lambda - \lambda^2 \Lambda \mathbf{f}^2 - m^2 \mathbf{a}^2 \right]. \quad (8)$$

⁶A careful consideration of the general successive dimensional reductions of Einstein's gravity from $D = 4$ to $D = 1$ is given in [26], [27]. For a review of the standard spherical and cylindrical reductions and solutions see [28].

⁷We suppose that the fields φ, a_i are dimensionless while $[t] = [r] = \text{L}$ and thus $[f_{ij}] = \text{L}^{-1}$, $[R] = [k_\nu] = [X] = [m^2] = \text{L}^{-2}$. For more details on our dimensions see Appendix.

⁸The expression for the determinant in Eq.(2) in terms of f_{ij} for $D = 4$ written in [19] contains also the term of the fourth order in the vector field. It is not difficult to see that in both spherical and cylindrical reductions (see the end of this Section) this term vanishes and thus Eq.(5) is valid. In fact, this is true in any dimension D .

These two-dimensional reductions are essentially simpler than their parent higher dimensional theories. In particular, we show that the massive vecton field theory can be transformed into a dilaton - scalaron gravity (DSG) model which is easier to analyze. Unfortunately, these DSG models and their further reductions to dimension one (static and cosmological reductions) are also essentially non-integrable. It is well known that the massless case, being a pure dilaton gravity, is classically integrable (see, e.g., [29] and reference therein). Having this in mind, we will attempt to find some additional integrals of motion in similar vecton models.

The next simplified theory is obtained in a *cylindrically symmetric* case. We consider here only $D = 3$ and $D = 4$. The general cylindrical reduction was discussed in detail in [25] and here we only summarize the main results. The most general cylindrical Lagrangian can be derived by applying the general Kaluza reduction to $D = 4$. The corresponding metric may be written as

$$ds_4^2 = (g_{ij} + \varphi \sigma_{mn} \varphi_i^m \varphi_j^n) dx^i dx^j + 2\varphi_{im} dx^i dy^m + \varphi \sigma_{mn} dy^m dy^n, \quad (9)$$

where $i, j = 0, 1$, $m, n = 2, 3$, all the metric coefficients depend only on the x -coordinates (t, r) , and $y^m = (\varphi, z)$ are coordinates on the two-dimensional cylinder (torus). Note that φ plays the role of the dilaton and σ_{mn} ($\det \sigma_{mn} = 1$) is the so-called σ -field. The reduction of the Einstein part of the four-dimensional Lagrangian, $\sqrt{-g_4} R_4$, can be written as:

$$\mathcal{L}_{4c}^{(2)} = \sqrt{-g} \left[\varphi R(g) + \frac{1}{2\varphi} (\nabla \varphi)^2 - \frac{\varphi}{4} \text{tr}(\nabla \sigma \sigma^{-1} \nabla \sigma \sigma^{-1}) - \frac{\varphi^2}{4} \sigma_{mn} \varphi_{ij}^m \varphi^{nij} \right], \quad (10)$$

where $\varphi_{ij}^m \equiv \partial_i \varphi_j^m - \partial_j \varphi_i^m$. The Abelian gauge fields φ_i^m are not propagating and their contribution is *usually neglected*. In [25] we proposed to take them into account by solving their equations of motion and writing the corresponding effective potential (similarly to what we are doing below in the spherically symmetric vecton gravity). Introducing a convenient parametrization for σ_{mn} ,

$$\sigma_{22} = e^\eta \cosh \xi, \quad \sigma_{33} = e^{-\eta} \cosh \xi, \quad \sigma_{23} = \sigma_{32} = \sinh \xi,$$

one can exclude the gauge fields φ_i^m and derive the effective action

$$\mathcal{L}_{\text{eff}}^{(2)} = \sqrt{-g} \left[\varphi R(g) + \frac{1}{2\varphi} (\nabla \varphi)^2 - \frac{\varphi}{2} [(\nabla \xi)^2 + (\cosh \xi)^2 (\nabla \eta)^2] + V_{\text{eff}}(\varphi, \xi, \eta) \right]. \quad (11)$$

Here the first three terms are standard, while the effective geometric potential,

$$V_{\text{eff}}(\varphi, \xi, \eta) = -\frac{\cosh \xi}{2\varphi^2} \left[Q_1^2 e^{-\eta} - 2Q_1 Q_2 \tanh \xi + Q_2^2 e^\eta \right], \quad (12)$$

depends on two new arbitrary real constants Q_m , which may be called ‘charges’ of the Abelian geometric gauge fields φ_i^m . This representation of the action is more convenient for writing the equations of motion, for further reductions to dimensions $(1+0)$, and $(0+1)$ as well as for analyzing special cases, such as $Q_1 Q_2 = 0$, $\xi \eta \equiv 0$.⁹

The static solutions of the theory (12) with $Q_1 Q_2 \neq 0$ have horizons while the exact solutions discussed in [25] for $Q_1 = Q_2 = 0$ and non-vanishing σ -fields ξ, η have no horizons at all, in accordance with the general theorem of papers [29]. An interesting special case can be obtained if we choose $Q_1 \neq 0$, $Q_2 = 0$, $\xi \equiv 0$. Then, Lagrangian (11) gives a standard dilaton gravity coupled to the scalar field η , with the potential depending both on the scalar field and dilaton. If $Q_1 \neq 0$, there exists a static solution with a horizon, which disappears when Q_1 vanishes. Of course, the horizon also exists in pure dilaton gravity, when $\xi = \eta = 0$.

⁹It is also closer to the original Einstein - Rosen equations for nonlinear gravitational waves, which can be obtained by putting $Q_1 = Q_2 = 0$ and $\xi \equiv 0$. When $Q_1 Q_2 \neq 0$, ξ and η cannot be identically zero.

In [29] we studied in some detail the models with the potentials independent of the scalar. In this paper, we demonstrate that interesting results can be derived in more general models with ‘separable’ potentials $V(\varphi, \psi) = v_1(\varphi)v_2(\psi)$. In particular, we show that one of the integrals of motion in the dilaton gravity coupled to massless scalars derived in [29] may exist also in some models with separable potentials that are of interest in the context of the present study.

We see that the general cylindrical action is a very complex two-dimensional theory and even its one-dimensional reductions are rather complex and in general not integrable. Adding the vecton sector does not make it simpler and more tractable. Much more tractable is the *three-dimensional* cylindrical space-time. The metric can be obtained by the obvious reduction of (9),

$$ds_3^2 = (g_{ij} + \varphi_i \varphi_j) dx^i dx^j + 2\varphi_i dx^i dy + \varphi dy^2, \quad (13)$$

and the corresponding Einstein Lagrangian is simply

$$\mathcal{L}_{3c}^{(2)} = \sqrt{-g} \varphi \{R(g) - \frac{\varphi}{4} \varphi_{ij} \varphi^{ij}\}, \quad \varphi_{ij} \equiv \varphi_{i,j} - \varphi_{j,i}. \quad (14)$$

Using the equation of motion for φ_{ij} and introducing the corresponding effective potential (see [25] and more general derivation below) we find the following two-dimensional dilaton gravity

$$\mathcal{L}_{\text{eff}}^{(2)} = \sqrt{-g} \{\phi R(g) - 8Q^2 \phi^{-3}\}, \quad \phi \equiv \sqrt{\varphi}. \quad (15)$$

As distinct from the cylindrical reduction of the four-dimensional pure Einstein theory corresponding to $V_{\text{eff}} = 0$ in (11), this theory has a horizon and can easily be integrated.

It is not difficult to include the vecton part into the cylindrically symmetric Lagrangians. In fact, the terms $-m^2 \varphi \mathbf{a}^2$, $X(\varphi, \mathbf{f}^2)$ in (5) are invariant and have the same form in any dimension. Therefore we can simply add the expressions (11), (14) to the gravitational Lagrangians also in the cylindrical case. Note however that there exist different cylindrical reductions of the vecton potential a_i . For example, unlike the spherical case, these fields may be nonzero for $i = 0, 1, 2$ and correspondingly $e_1 \equiv f_{01} \equiv \partial_0 a_1 - \partial_1 a_0 \neq 0$, $e_2 \equiv f_{02} \equiv \partial_0 a_2 \neq 0$, $h_3 \equiv f_{12} \equiv \partial_1 a_2 \neq 0$.¹⁰ We see that the component $a_2 \equiv a_\varphi$ of the vector field a_i behaves like an additional scalar field and, in addition to the two-dimensional vector field (a_0, a_1) , we have up to three scalar matter fields, a_2, ξ, η , with a rather complex interaction to the dilaton gravity. This means that analyzing cylindrical solutions is more difficult than that of the spherical ones. Our consideration below is applicable to the simplest case of two potentials a_0, a_1 with one field f_{01} .

3 Vector - scalar duality in two-dimensional vecton theory

In the dimension $D = 2$ all fields (vector, spinor, ...) are practically equivalent to scalar ones. Such equivalence is widely known for massless Abelian gauge fields (see, e.g., [29] - [32] and references therein). The aim of this Section is to establish a standard map of massive Abelian vector fields to scalar fields. We do not attempt to present the most general results in this direction and therefore restrict our consideration to a DVG coupled to scalar ‘matter’ fields. This includes all the models of the previous sections. Suppose that in place of the standard Abelian gauge field term, $X(\varphi, \psi) \mathbf{f}^2$, the Lagrangian contains a more general coupling of the gauge field $f_{ij} = \partial_i a_j - \partial_j a_i$ to the dilaton and scalar fields, $X(\varphi, \psi; \mathbf{f}^2)$, where $\mathbf{f}^2 \equiv f_{ij} f^{ij}$. Using the Weyl transformation we write a fairly general two-dimensional Lagrangian in the form

$$\mathcal{L}^{(2)} = \sqrt{-g} \left[\varphi R + V(\varphi, \psi) + X(\varphi, \psi; \mathbf{f}^2) + Z_a(\varphi) \mathbf{a}^2 + \sum Z(\varphi, \psi) (\nabla \psi)^2 \right], \quad (16)$$

¹⁰Here $e_i \equiv a_{0i}$, $h_i \equiv \varepsilon_{ijk} a_{jk}$. In the diagonal metric $g_{ij} = g_i \delta_{ij}$ the fourth-order term in the determinant from Eq.(2) is proportional to $(e_i h_i)^2 \exp \sum 2g_k$ and is seen to vanish. This argument obviously works in any dimension.

where the kinetic dilaton term $\sim \varphi^{-1}(\nabla\varphi)^2$ is now absent while V and Z terms represent potentials and kinetic terms of scalar fields (like those in (11), (12)). We need not specify the number of the scalar matter fields and therefore omit the summation indices for the fields ψ and their Z -functions. For the fields, having positive kinetic energy, all the Z -functions in (16) are negative and usually proportional to φ ; we call them ‘normal matter’ fields.

In [30], we considered the massless case, $Z_a \equiv 0$, and proposed to use instead of (16) the effective Lagrangian not containing the Abelian gauge fields (for a detailed proof see [31], [32]). In this paper, we extend this approach to the massive vecton and show that the vector field can be effectively replaced by a scalar one (*scalaron*) denoted by $q(t, r)$. This means that the equations of motion defined by Lagrangian (16) can be obtained from the Lagrangian,

$$\mathcal{L}_{\text{eff}}^{(2)} = \sqrt{-g} \left[\varphi R + V(\varphi, \psi) + X_{\text{eff}}(\varphi, \psi, q) + Z_q(\varphi)(\nabla q)^2 + \sum Z(\varphi, \psi)(\nabla\psi)^2 \right], \quad (17)$$

by applying a formally invertible transformation from the vecton fields a_i, f_{ij} to the scalaron fields $\partial_i q, q$. This can be constructed in a complete analogy to the massless case.

First, consider the equations for the vecton fields derived from Lagrangian (16):

$$2\partial_j(\sqrt{-g}X'f^{ij}) = Z_a(\varphi)\sqrt{-g}a^i, \quad X' \equiv X'(\varphi, \psi; \mathbf{f}^2) \equiv \frac{\partial X}{\partial \mathbf{f}^2}, \quad (18)$$

At this point, it is convenient to rewrite the Lagrangian and equations in the LC coordinates,

$$ds^2 = -4h(u, v) du dv, \quad \sqrt{-g} = 2h, \quad f_{uv} \equiv a_{u,v} - a_{v,u}, \quad -2\mathbf{f}^2 = (f_{uv}/h)^2. \quad (19)$$

Before we pass to the LC Lagrangian we should derive the energy and momentum constraints using the general coordinates. Leaving only one extra field ψ we have (see, e.g. [18], [29] - [34]):

$$h\partial_i(\partial_i\varphi/h) = Z_a(\varphi)a_i^2 + Z(\varphi, \psi)(\partial_i\psi)^2, \quad i = u, v. \quad (20)$$

The other equations of motion can be derived from the LC transformed Lagrangian,

$$\frac{1}{2}\mathcal{L}^{(2)} = \varphi\partial_u\partial_v\ln|h| + h[V(\varphi, \psi) + X(\varphi, \psi, \mathbf{f}^2)] - Z_a(\varphi)a_u a_v - Z(\varphi, \psi)\partial_u\psi\partial_v\psi, \quad (21)$$

simply by variations in $h, \psi, \mathbf{f}^2, \varphi$. The most important equations are the following:

$$\partial_u\partial_v\varphi + h\left[X(\varphi, \psi, \mathbf{f}^2) + h\frac{\partial X}{\partial \mathbf{f}^2}\frac{\partial \mathbf{f}^2}{\partial h}\right] \equiv \partial_u\partial_v\varphi + h\left[X - 2\mathbf{f}^2\frac{\partial X}{\partial \mathbf{f}^2}\right] = 0, \quad (22)$$

which allows us to define the effective potential $X_{\text{eff}}(\varphi, \psi, q)$, and the vecton equations,

$$\partial_u(h^{-1}X'f_{uv}) = -Z_a(\varphi)a_u, \quad \partial_v(h^{-1}X'f_{uv}) = Z_a(\varphi)a_v, \quad (23)$$

that are equivalent to (18). Now we transform them into a standard scalar equation for a new scalar field $q(u, v)$, which can be defined both in the LC and in general coordinates:

$$q(u, v) \equiv h^{-1}X'f_{uv}, \quad 2\sqrt{-g}f^{ij}X' \equiv \varepsilon^{ij}q, \quad i, j = 0, 1. \quad (24)$$

It is not difficult to understand that so defined scalar q is also invariant under the Weyl transformation (6). With this definition, we may regard \mathbf{f}^2 as a function of q , which can be derived by solving (24). Denoting the solution by $z(q)$ and expressing f_{uv}/h in terms of \mathbf{f}^2 with the aid of (19) we find an important relation:

$$2z(q) = -(q/\bar{X}')^2, \quad \bar{X}' \equiv \frac{\partial}{\partial z(q)}X[\varphi, \psi; z(q)]. \quad (25)$$

Returning to (22), we see that the expression in the square brackets, with \mathbf{f}^2 replaced by $z(q)$, is in fact our *effective potential* $X_{\text{eff}}(\varphi, \psi, q)$. There exist several expression for it that allow one to derive the same equations of motion for the fields h, φ, ψ, q . In [31], [32] we obtained two equivalent expressions for X_{eff} by using (22) and (25)):

$$X_{\text{eff}}(\varphi, \psi, q) = X[\varphi, \psi; z(q)] - 2z(q)\bar{X}' = X[\varphi, \psi; z(q)] + q^2/\bar{X}'[\varphi, \psi; z(q)]. \quad (26)$$

From (25) and (26) we also derived the most compact and beautiful representation:

$$X_{\text{eff}}(\varphi, \psi, q) = X(\varphi, \psi; z) + q\sqrt{-2z}, \quad (27)$$

where we intentionally write z instead of $z(q)$. The reason is that

$$\partial_q X_{\text{eff}}(\varphi, \psi, q) = (\bar{X}' - q/\sqrt{-2z}) \partial_q z(q) + \sqrt{-2z} = \sqrt{-2z}, \quad (28)$$

as follows from Eq.(25). Now it is easy to write the equation for q . Using (23), (24) we find:

$$a_u(u, v) = -Z_a^{-1}(\varphi) \partial_u q(u, v), \quad a_v(u, v) = Z_a^{-1}(\varphi) \partial_v q(u, v), \quad (29)$$

$$\partial_v(Z_a^{-1} \partial_u q) + \partial_u(Z_a^{-1} \partial_v q) = -f_{uv}(q) = -hq/\bar{X}' = -h\partial_q X_{\text{eff}}(\varphi, \psi, q) \quad (30)$$

The last equation for $q(u, v)$ can be derived from Lagrangian (17) by taking $Z_q = Z_a^{-1}$. Usually, $Z_a = -m^2\varphi$ and then the scalaron kinetic term has the unusual form,

$$Z_q(\nabla q)^2 = Z_a^{-1}(\nabla q)^2 = (\nabla q)^2/(-m^2\varphi). \quad (31)$$

It is not difficult to derive all the equation of motion using effective Lagrangian (17) with X_{eff} and Z_q given by (27), (31), and we list them here for further reference. In addition to the relations obtain above, one has to use the following easily checked identities:

$$\partial_h X_{\text{eff}}(\varphi, \psi, q) = 0, \quad \partial_\varphi X_{\text{eff}} = \partial_\varphi X(\varphi, \psi, q), \quad \partial_\psi X_{\text{eff}} = \partial_\psi X(\varphi, \psi, q).$$

The first identity immediately follows from our definitions. In fact, it is a characteristic property of X_{eff} and thus can serve to define it. Two other identities are derived like Eq.(28).

Let us define the total effective potential,

$$U(\varphi, \psi, q) \equiv V(\varphi, \psi) + X_{\text{eff}}(\varphi, \psi, q), \quad (32)$$

and first write the LC version of the effective Lagrangian and of the constraints

$$\frac{1}{2}\mathcal{L}_{\text{eff}}^{(2)} = \varphi \partial_u \partial_v \ln |h| + hU(\varphi, \psi, q) - Z_q(\varphi) \partial_u q \partial_v q - Z(\varphi, \psi) \partial_u \psi \partial_v \psi, \quad (33)$$

$$h \partial_i (\partial_i \varphi / h) - Z_q(\varphi) (\partial_i q)^2 - Z(\varphi, \psi) (\partial_i \psi)^2 = 0. \quad i = u, v. \quad (34)$$

The other equations are the same as used in the dilaton gravity coupled to scalars:¹¹

$$\partial_u \partial_v \varphi + hU(\varphi, \psi, q) = 0, \quad (35)$$

$$\partial_u(Z_q \partial_v q) + \partial_v(Z_q \partial_u q) + h \partial_q U(\varphi, \psi, q) = 0, \quad (36)$$

$$\partial_u(Z \partial_v \psi) + \partial_v(Z \partial_u \psi) = \partial_\psi [-hU + Z(\varphi, \psi) \partial_u \psi \partial_v \psi], \quad (37)$$

$$\partial_u \partial_v \ln |h| = \partial_\varphi [-hU + Z_q(\varphi) \partial_u q \partial_v q + Z(\varphi, \psi) \partial_u \psi \partial_v \psi]. \quad (38)$$

¹¹Below, we also use notation $\ln |h| \equiv F$, $h \equiv \epsilon \exp F$, where $\epsilon = +1$ for cosmological and -1 for static solutions.

One of the last three equations is satisfied by solutions of the remaining five. The energy constraint, which is the sum of the two constraints, plays a special role in the ADM Hamiltonian formulation [35]. We do not discuss here these subtleties that will be clarified in considering the one-dimensional reductions of this two-dimensional quasi-linear constrained system of the second order. The structure of this system allows one to find some integrable classes using simplifying assumptions about the potentials U and Z . For some multi-exponential potentials U and constant potentials Z , there exist integrable systems related to Liouville and Toda-Liouville ones (see [36], [37], [33]). The pure Liouville case was completely solved but, for the Toda-Liouville, it is difficult to find an exact analytical solution of the two-dimensional constraints, even in the simplest $u_1 \oplus su_2$ case. This problem is easily solved in the one-dimensional (static or cosmological) case.

For the spherical $D = 3$ and $D = 4$ Lagrangians in the Weyl frame (see (8) and (7), resp.), it is easy to derive the effective potential $\hat{U} \equiv U/w(\varphi)$ and thus to find potential (32):

$$U(\varphi, q) = \hat{U}(\varphi, q) = -2\Lambda\varphi [1 + q^2/4\lambda^2\Lambda^2\varphi^2], \quad D = 3, \quad (39)$$

$$U(\varphi, q) = \sqrt{\varphi}\hat{U}(\varphi, q) = -2\Lambda\varphi [1 + q^2/\lambda^2\Lambda^2\varphi^2]^{\frac{1}{2}} + 2k, \quad D = 4. \quad (40)$$

We remind that the potential Z_q is D -independent, $Z_q = -1/m^2\varphi$, see Eq.(31).

As was argued in [18], even the one-dimensional cosmological reduction of the pure scalaron theory with the potential (39) is not integrable. Thus we concentrate on searching for approximate potentials that allow us to find a wide enough class of analytic solutions to reconstruct exact ones by iterations. For instance, potential (40) for large q is asymptotically very simple, in the standard frame we have $U = -2q/\lambda + \dots$. As will be shown shortly, this asymptotic formula is valid in dimensions $D > 3$. On the other hand, the behavior of the q -dependent part of U for small q is given by (39), if we multiply the q^2 term by $(D - 2)$. This universality allows us in what follows to regard (39), (40) the generic scalaron potentials.

4 Cosmological and static reductions of scalaron theory

In this paper, we consider only the simplest reductions of the two-dimensional theory ignoring one-dimensional waves studied in our previous work [30]-[33], [37]. The simplest reduced gauge fixed equations can be directly derived by supposing that in the LC equations (33) - (38) the fields h, φ, q, ψ depend on one variable, which we denote $\tau \equiv (u + v)$. For the cosmological solutions this variable is identified with the time variable, $\tau = t$, while for the static states, including black holes (BH), it is the space variable, $\tau = r$. The only difference between the cosmological and static solutions is in the sign of the metric, $h_c > 0$, $h_s < 0$.

In our study of black holes and cosmologies we use the general diagonal metric,

$$ds_2^2 = e^{2\alpha(t,r)} dr^2 - e^{2\gamma(t,r)} dt^2. \quad (41)$$

Then, the static and cosmological reductions of our two-dimensional vector theory (17) can be presented by the Lagrangian (with the standard $Z(\varphi, \psi) = -\varphi$)

$$\epsilon \mathcal{L}_v^{(1)} = e^{\epsilon(\alpha-\gamma)} \varphi \left[\dot{\psi}^2 - 2\dot{\alpha}_\epsilon \frac{\dot{\varphi}}{\varphi} - (1-\nu) \left(\frac{\dot{\varphi}}{\varphi} \right)^2 \right] - e^{\epsilon(\gamma-\alpha)} \mu \varphi a_\epsilon^2 + \epsilon e^{\alpha+\gamma} \left[V + X(\mathbf{f}^2) \right]. \quad (42)$$

Here we omit a possible dependence of V and X on φ and ψ , denote $Z_a \equiv -\mu\varphi \equiv m^2$, and $\epsilon = \pm$. All the fields depend on $\tau = t$ ($\epsilon = +$) or on $\tau = r$ ($\epsilon = -$). Finally,

$$a_+ = a_1(\tau), \quad a_- = a_0(\tau), \quad \alpha_+ \equiv \alpha, \quad \alpha_- \equiv \gamma, \quad \dot{\alpha}_\epsilon = \frac{d}{d\tau} \alpha_\epsilon, \quad \dot{a} = \frac{d}{d\tau} a.$$

We see that the cosmological and static Lagrangians essentially coincide, the only difference being in the sign of the potential term and of the metric exponents as well. If $\epsilon = \pm$, the multiplier of the kinetic term, $l_\epsilon \equiv e^{\alpha_\mp}$, is obviously a Lagrange multiplier varying of which produces the constraint equation, which is equivalent to vanishing of the Hamiltonian. In view of the implicit dependence of \mathbf{f}^2 on l_ϵ , it is much more convenient to first employ the canonical formulation and then identify a proper Lagrange multiplier. Alternatively, one can follow the steps made in previous Section but the canonical approach is simpler both technically and conceptually.

Let us construct the effective Hamiltonian and Lagrangian of the vecton sub-Lagrangian,

$$\mathcal{L}_a \equiv h X(\mathbf{f}^2) - \tilde{\mu} \varphi a^2; \quad \tilde{\mu} = e^{\epsilon(\gamma-\alpha)} \mu, \quad h = \epsilon e^{\alpha+\gamma}. \quad (43)$$

Remembering our previous results we define the variable y and derive the momentum p_a ,

$$\frac{1}{2} \lambda^2 \mathbf{f}^2 = -\lambda^2 \left(\frac{\dot{a}}{h} \right)^2 = -y^2, \quad \dot{a} \equiv \frac{1}{\lambda} h y, \quad p_a \equiv \frac{\partial \mathcal{L}_a}{\partial \dot{a}} = \lambda \frac{\partial X}{\partial y}, \quad (44)$$

where the last equation implicitly defines $y(p_a)$. This allows us to write the partial Hamiltonian,

$$\mathcal{H}_a(p_a, a) \equiv p_a \dot{a} - \mathcal{L}_a = -h \left[X - y \frac{\partial X}{\partial y} \right] + \tilde{\mu} \varphi a^2 \equiv -h X_{\text{eff}} + \tilde{\mu} \varphi a^2. \quad (45)$$

Here X_{eff} essentially coincides with the effective potential derived in the previous section but now depends on φ, ψ and $y(p_a)$. Applying the simplest possible canonical transformation,

$$p_a \Rightarrow -2q, \quad a \Rightarrow p/2, \quad X_{\text{eff}}(p_a) \Rightarrow X_{\text{eff}}(-2q), \quad (46)$$

we see that the effective potentials (and the partial Lagrangians) obtained here and in Section 3 are identical. Indeed, applying the canonical transformation to Hamiltonian (45), we find the partial Hamiltonian of the scalaron, $\mathcal{H}_s(p, q)$, and the corresponding Lagrangian,

$$\mathcal{L}_s(p, q) \equiv p \dot{q} - \mathcal{H}_s(p, q) = h X_{\text{eff}} + e^{\epsilon(\alpha-\gamma)} \frac{\dot{q}^2}{\mu \varphi}, \quad (47)$$

which can now be inserted into the complete Lagrangian instead of \mathcal{L}_a .

Thus we have the complete DSG Lagrangian, now denoting $l_\epsilon \equiv \exp(\alpha_{-\epsilon} - \alpha_{+\epsilon})$,

$$\epsilon \mathcal{L}_q^{(1)} = l_\epsilon^{-1} \left[\varphi \dot{\psi}^2 - 2\dot{\alpha}_\epsilon \dot{\varphi} - (1-\nu) \frac{\dot{\varphi}^2}{\varphi} + \frac{\dot{q}^2}{m^2 \varphi} \right] + l_\epsilon \epsilon e^{2\alpha_\epsilon} U(\varphi, \psi, q), \quad (48)$$

where we also use notation (32) and $\mu = m^2$. This form is more natural than (42). First, the dependence on the Lagrangian multiplier l_ϵ is simple and standard, the kinetic part is quadratic in generalized velocities and can be made diagonal by a redefinition of the Lagrangian multiplier and velocities. In addition, we are free to make a convenient gauge choice and to choose the Weyl frame. For example, by making the shift $\alpha_\epsilon \Rightarrow \alpha_\epsilon - (1-\nu) \ln \sqrt{\varphi}$ and redefining the potential by $U \Rightarrow \varphi^{\nu-1} U$ we remove the third term in (48) and obtain the Lagrangian in Weyl's frame.¹² Then we can redefine $l_\epsilon \varphi \equiv \bar{l}_\epsilon$, introduce the new field $\xi \equiv \varphi^2$ and finally rewrite (48) in a simpler form,

$$\epsilon \mathcal{L}_q^{(1)} = \bar{l}_\epsilon^{-1} [\xi \dot{\psi}^2 + m^{-2} \dot{q}^2 - \dot{\xi} \dot{\alpha}_\epsilon] + \bar{l}_\epsilon \epsilon e^{2\alpha_\epsilon} \xi^{\nu/2-1} U(\sqrt{\xi}, q, \psi). \quad (49)$$

¹²After this transformation, the theory defined by constrained Lagrangian (48) can be compared to the one-dimensional reduction of the theory defined by equations (33) - (38): we first derive the constraint, choose the LC metric by putting $l_\epsilon = 1$, and then derive all the equations of motion. Note that the general one-dimensional Lagrangian (48) gives more freedom in simplifying equations. For example, if there are only standard fields ψ , we may choose $l_\epsilon \equiv l_\epsilon \varphi$, $\tilde{\xi} \equiv \ln \varphi$ and then there will be no need to use Weyl's frame for simplifying the kinetic terms.

Before applying it to studies of cosmologies and horizons in the scalaron theory we discuss the effective scalaron potential, corresponding to the X -potential (5) in more detail.

Using notation (44) it is not difficult to find that $y(p_a)$ can be derived from the equation

$$y = x(1 - y^2)^{1-\nu}, \quad x \equiv q/(-2\nu\lambda\Lambda\varphi), \quad (50)$$

where we choose the sign of q so that $x > 0$ and therefore $y > 0$. This is a polynomial equation of order $(D - 3)$ for y^2 if D is even, and of order $2(D - 3)$ for y if D is odd. This observation is not very useful if $D > 4$, but some algebraic expressions for $y(x)$ can be written if $D = 5, 6$. They are rather complex and difficult to use. However, it is not difficult to find good approximations for the solutions for small and large values of x :

$$y = x[1 - (1 - \nu)x^2 + \dots]; \quad 1 - y^2 = |x|^{-\sigma} - (\sigma/2)|x|^{-2\sigma} + \dots, \quad \sigma \equiv 1/(1 - \nu). \quad (51)$$

Both expansions are applicable to all $\nu > 1$, the first is exact for $\nu = 1$ as $y \equiv x$.

To analyze the behavior of $X_{\text{eff}}(x)$ let us rewrite it in terms of the solution $y(x)$ of Eq.(50)

$$X_{\text{eff}} = X - y \frac{\partial X}{\partial y} = -2\Lambda\varphi \frac{x}{y} \left[1 - (1 - 2\nu)y^2 \right]. \quad (52)$$

This expression is convenient for deriving some exact solutions as well as for discussing their general properties for arbitrary D . If $D = 3$, then $y = x$, and we directly find the effective potential (39). For $D = 4$, $x/y = \sqrt{1 + x^2}$, and thus the potential coincides with (40). An interesting feature of this effective potential is its invariance under rotations in the plane $(q, \lambda\Lambda\varphi)$. This invariance exists only in $D = 4$. Unfortunately, the kinetic part does not respect this symmetry for any gauge choice. A different, simpler symmetry has the effective potential in $D = 3$, which is equal to (39) divided by $\varphi \equiv \sqrt{\xi}$. Then Lagrangian (49) of the pure scalaron gravity, with $h \equiv \epsilon e^{2\alpha\epsilon}$ and $\psi \equiv 0$, has a very simple form,

$$\epsilon\mathcal{L}_q^{(1)} = \bar{l}_\epsilon^{-1} [m^{-2}\dot{q}^2 - \dot{\xi}\dot{\alpha}_\epsilon] - \bar{l}_\epsilon 2\Lambda h [1 + q^2/4\lambda^2\Lambda^2\xi], \quad (53)$$

in which it is easy to find the scale symmetry $q \mapsto Cq$, $\xi \mapsto C^2\xi$, $h \mapsto C^2h$. Such a symmetry itself does not allow to obtain an additional integral of motion. At most, it may signal of existence a special solution with a scaling-type dependence on q, ξ, h , but this is not our primary concern. Below, we will show that, in fact, the theory (53) has an additional integral, but first we should complete our discussion of the general properties of the theories (49).

Applying asymptotic approximations (51) to (52) we prove the universality mentioned above:

$$X_{\text{eff}} = -2\Lambda\varphi [1 + q^2/4\nu\lambda^2\Lambda^2\varphi^2 + O(x^4)], \quad X_{\text{eff}} = -2\sqrt{q^2/\lambda^2} + O(|x|^{-\sigma}). \quad (54)$$

The small q approximations is exact for $\nu = 1$, the large q approximation is independent both of φ and $\nu > 1$, and the general structure of X_{eff} is defined by a positive regular function, $v_\nu(x)$,

$$X_{\text{eff}}/(-2\Lambda\varphi) \equiv v_\nu(x); \quad v_\nu(0) = 1; \quad v_\nu(x) \simeq 2\nu x, \quad x \rightarrow \infty. \quad (55)$$

Thus the effective potential in Lagrangian (49) is

$$U_e(\varphi, x) \equiv \xi^{\nu/2-1} U(\sqrt{\xi}, q, \psi) = \varphi^{\nu-2} U = -2\Lambda\varphi^{\nu-1} v_\nu(x) + k_\nu \varphi^{-(1+\nu)}, \quad (56)$$

and the exact expressions for $\nu = 1, 1/2$ are given in (39), (40).

Let us discuss properties of $v_\nu(x)$, which can be considered as the function of $q^2/\varphi^2 \equiv q^2/\xi$ (it is expressed in terms of $y(x)$ by Eq.(52)). Using (50) and (51) it is easy to find that

$$\frac{dy}{dx} = \frac{y}{x} \frac{1 - y^2}{1 + (1 - 2\nu)y^2} > 0, \quad y'(0) = 1, \quad y'(\infty) = 0,$$

$$v'_\nu(x) = 2\nu y(x); \quad v'_\nu(x) > 0, \quad v''_\nu(x) > 0. \quad (57)$$

Thus $v_\nu(x)$ is monotonic concave function having simple expansions for $x \ll 1$, $x \gg 1$:

$$v_\nu(x) = 1 + \nu x^2 + O(x^4); \quad v_\nu(x) = 2\nu x \left[1 + \frac{1-\nu}{2\nu} x^{-\sigma} + O(x^{-2\sigma}) \right]. \quad (58)$$

With such a simple and regular potential U_e , one might expect that at least qualitative behavior of the solutions of the theory (49) could be analyzed for small and large values of x . This is true if the theory is integrable. But it is not integrable, even in the simplest $D = 3$ case when $v_1(x) = 1 + x^2$, $k_1 = 0$ and thus $U_e = -2\Lambda(1 + x^2)$ is linear in q^2/ξ . The integrability of this and of similar but more general theories is discussed in next Section and in a separate paper. Here we consider the scalaron canonical equations and find one additional integral of motion.

Consider the general pure scalaron theory in which we neglect the curvature term $k_\nu \varphi^{-(1+\nu)}$:

$$\mathcal{L}_q^{(1)} = \bar{l}_\epsilon^{-1} [\mu^{-1} \dot{q}^2 - \dot{\xi} \dot{\alpha}_\epsilon] - \bar{l}_\epsilon 2\Lambda h \varphi^{\nu-1} v_\nu(x), \quad \mu \equiv m^2. \quad (59)$$

We did not yet fix the gauge, \bar{l}_ϵ is arbitrary. Varying in it we get the vanishing Hamiltonian,¹³

$$\mathcal{H}_q^{(1)} \equiv \bar{l}_\epsilon H_q^{(1)} = \bar{l}_\epsilon [\mu p_q^2/4 - p_\alpha p_\xi + 2\Lambda h \varphi^{\nu-1} v_\nu(x)] = 0, \quad (60)$$

where we used the definition of the canonical momenta (here and below, α_ϵ is replaced by α),

$$\dot{q} = \bar{l}_\epsilon \mu p_q/2, \quad \dot{\xi} = -\bar{l}_\epsilon p_\alpha, \quad \dot{\alpha} = -\bar{l}_\epsilon p_\xi. \quad (61)$$

We may use the parametrization invariance of the theory to choose a most convenient evolution parameter τ , for example, $d\bar{\tau} \equiv \bar{l}_\epsilon d\tau = \dot{\xi} d\tau = d\xi$.¹⁴ Such a parametrization is equivalent to gauge fixing plus a coordinate transformation. The standard LC gauge choice in our present notation is $l_\epsilon = 1$, we mostly use the gauge $l_\epsilon \varphi \equiv \bar{l}_\epsilon = 1$ and keep the dot notation for $d/d\tau$. To further simplify the equations, consider the $\nu = 1$ case and denote $2\Lambda h v_1(x) \equiv h\omega(q^2/\xi)$. With this notation and with $\bar{l}_\epsilon = 1$, the Hamiltonian system is given by equations (61) plus

$$\dot{p}_q = -2h\omega'(q^2/\xi) q/\xi, \quad \dot{p}_\xi = h\omega'(q^2/\xi) q^2/\xi^2, \quad \dot{p}_\alpha = -2h\omega. \quad (62)$$

It is easy to see that this Hamiltonian system has two integrals of motion. The first is the Hamiltonian (60). The second one follows from the obvious relation $q\dot{p}_q + 2\xi\dot{p}_\xi = 0$:

$$d/d\tau (q p_q + 2\xi p_\xi) = q\dot{p}_q + 2\xi\dot{p}_\xi = -2h\omega = \dot{p}_\alpha, \quad (63)$$

where we used (61), Hamiltonian constraint (60), and the last equation in (62). Integrating this relation gives the desired second integral. Thus we have two integrals:

$$\mu p_q^2 - 4p_\alpha p_\xi + 4h\omega = 0; \quad q p_q + 2\xi p_\xi - p_\alpha = c_0. \quad (64)$$

Knowledge of these integrals does not allow us to integrate our system but it is helpful in analyzing its properties. One may hope that, for some particular potentials ω , a third integral can be found. For these reasons we call such systems *partially integrable*. The present observation suggests a more general approach to constructing system with additional integrals presented in next Section.

¹³Alternatively, we can use the standard Legendre transformation and write the Hamiltonian theory in the Lagrangian form, $\mathcal{L}(q, \dot{q}) = p\dot{q} - lH(p, q)$, which is most convenient for systems with constraints (see, e.g., [38]).

¹⁴This choice is locally possible and is good as far as $\dot{\xi} \neq 0$. Here we ignore such subtleties.

5 Integrals and integrability in simple cases

Here we consider a general DGS with one scalar ψ that may be a standard field or the scalaron:

$$\mathcal{L}_{\text{dgs}}^{(2)} = \sqrt{-g} \left[\varphi R + Z(\varphi)(\nabla\psi)^2 + V(\varphi, \psi) \right]. \quad (65)$$

For the scalaron we have $\psi = q$, $Z = Z_q = -1/(m^2\varphi)$ and the potentials are given above. For the standard scalar $Z_\psi \sim -\varphi$, but some of the results presented below are applicable to more general Z -functions. In our notation, negative signs of Z correspond to positive kinetic energies of the scalar fields but our classical consideration is fully applicable to both signs. The general model (65) with a general potential V is not integrable in any sense. One of the strong obstructions to integrability is the dependence of Z on φ , and the usual simplifying assumption is that the Z -functions are independent of φ . With this restriction, there exists a class of ‘multi - exponential’ potentials, for which the DGS theories with any number of scalar fields can be reduced to the Toda - Liouville systems and exactly solved.¹⁵ For their static - cosmological reductions, analytic solutions were explicitly derived. Here we try to expand this class of the models.

First, consider the case of the massless vecton. In the pure scalaron DG we have, in the massless limit, $h\partial_i(\partial_i\varphi/h) = 0$, from which we find $h(u, v) = h(\tau)$, $\varphi(u, v) = \varphi(\tau)$, where $\tau = a(u) + b(v)$. This means that the theory is automatically reduced to dimension one and we can use the Hamiltonian formulation of Section 4 (in the LC gauge). It is clear q is a constant, $q \equiv q_0$, as can also be seen from (23), (24). Now, using the above definitions of the momenta p_α, p_ξ , constraint $p_\alpha p_\xi = h\omega(\xi)$, and equation $\dot{p}_\alpha = -2h\omega(\xi)$, all of which are valid for arbitrary potential $\omega(\xi)$, one can explicitly solve the equations of motion:

$$\dot{p}_\alpha = -2h\omega = -2p_\alpha p_\xi \quad \Rightarrow \quad 2\dot{\alpha}p_\alpha = \dot{p}_\alpha \quad \Rightarrow \quad -\dot{\xi} \equiv p_\alpha = -c_1 h \quad \Rightarrow \quad c_1 \dot{h} = -2h\omega, \quad (66)$$

where c_1 is the integration constant. Then it follows the well known expression for h ,

$$c_1^2 \frac{dh}{d\xi} = 2\omega(\xi), \quad c_1^2 h = 2 \int d\xi \omega(\xi) = 2 \int d\varphi V(\varphi) \equiv N(\varphi) - N(\varphi_0), \quad (67)$$

if we recall that the transformation of ξ to φ induces the transformation of the potential, so that $d\xi \omega(\xi) = d\varphi V(\varphi)$. Thus we find the explicit general solution of the general dilaton gravity, if we in addition solve the equation $\dot{\xi} = c_1 h(\xi)$ defining the τ -dependence. Now, in our special case of the dilaton - scalaron gravity corresponding to the massless vecton, the only trace of the vecton is the dependence of the potential on q_0 , the ‘charge’ of the massless vector field, and on other parameters defining the theory. The general solution depends on the free independent parameters, (c_1, φ_0, τ_0) ,¹⁶ which can be hidden by rescaling of h and φ and by shifting τ . Then the ‘portrait’ of the solution is a curve in the (h, φ) plane significantly depending on q_0 . The most important feature of this portrait is the structure of the set of the horizons. For us, the most interesting is the dependence of this portraits on the parameters of the parent DSG: $q_0, \Lambda, \lambda, \nu$. Characteristic properties of the portraits are: the number and structure of the horizons (simple, degenerate, singular) and the behavior of the solutions near horizons and singularities. A more interesting portrait of an integrable DGS is described in Appendix.

For DGS (65), we can derive some convergent expansions of exact solutions near the horizons by applying the general approach proposed in [34]. Here we formulate its most compact form,

¹⁵This class includes all previously considered integrable two-dimensional DGS, which are reviewed in [36] (see also [39]). The first DGS of the Liouville type (‘bi Liouville’), which generalizes the so-called Jackiw [40] and CGHS [41] models, was proposed and solved in papers [29], the results of which are essentially generalized here.

¹⁶The physical parameters are defined by the equation for horizons, $N(\varphi) - N(\varphi_0) = 0$. When this equation has a unique non-degenerate solution, as in the case of the Schwarzschild black hole, $N(\varphi_0)$ can be related to its ‘mass’; the case of many solutions, φ_n , the physical interpretation is much reacher, see, e.g. [26], [42].

which is convenient to use in general DGS with arbitrary Z , V ($V_\psi \neq 0$), and any number of scalars. For the sake of generality, we do not explicitly use the canonical formalism and thus somewhat change notation to the one resembling notation of [29]. We will also use the LC gauge and Weyl's frame having in mind that Eqs.(48)-(49) show how to return to the standard frame and general gauge. Thus the system of one-dimensional equations can be obtained by dimensional reduction of (34)-(38) if we suppose that all unknown functions depend on one variable $\tau = u + v$. Denoting $F \equiv \ln h$ and introducing the new momentum-like variables χ, η, ρ ,

$$\dot{\varphi} = \chi, \quad Z(\varphi) \dot{\psi} = \eta, \quad Z(\varphi) \dot{F} = \rho, \quad (68)$$

we rewrite the main dynamical equations and the constraint as

$$\dot{\chi} + hV = 0, \quad 2\dot{\eta} + hV_\psi = 0, \quad \dot{\rho} + h(ZV)_\varphi = 0, \quad \chi\rho + hZV + \eta^2 = 0, \quad (69)$$

where the lower indices φ , and ψ denote the corresponding partial derivatives. It is well known that the second or the third equation can be omitted¹⁷ and actually we have two independent second order differential equations and one constraint which is the first order equation. Therefore, to completely solve this system we must search for two additional constraints canonically commuting with the Hamiltonian and with each other.¹⁸

Now, recalling the previous section we change the variable τ to ξ that is defined by the relations: $\chi d\tau = d\varphi \equiv Z d\xi$. Then we rewrite equations (68)-(69) denoting the derivative $d/d\xi$ by the prime and introducing useful notation: $U(\xi, \psi) \equiv Z(\varphi)V(\varphi, \psi)$, $H(\xi) \equiv h/\chi$, and $G(\xi) \equiv \eta/\chi$. The main independent equations for χ, η, ψ, H now have a very compact form:

$$\psi' = G, \quad H' = -G^2 H, \quad \chi' + UH = 0, \quad 2\eta' + U_\psi H = 0. \quad (70)$$

The extended system contains two equations for ρ (see (68)-(69)):

$$\chi F' - \rho = 0, \quad \rho' + U_\xi H = 0, \quad \chi G' = UH(G - U_\psi/2U), \quad (71)$$

where we add the explicit equation for G , which may be used instead of the last equation in (70).

The system (70) is most convenient for deriving the solutions near horizons and in asymptotic regions and for studying their general properties. For example, a very important property of its solution is that $(\ln H)' = -G^2 < 0$. This property does not depend on the potential and is true for any number of scalar fields provided that their Z -functions are negative, as was first shown in [22]. Indeed, in this case the constraint equation can be written as:

$$\Phi' \equiv (\ln H)' = -Z_0 \sum_{n=0}^N Z_n^{-1}(\xi) (\eta_n/\chi)^2, \quad \eta_n \equiv Z_n \dot{\psi} \equiv \chi \psi'. \quad (72)$$

For normal fields $Z_n < 0$ for all n , for some anomalous fields, like scalaron corresponding to tachyonic vevton, Z may be positive and then the sign of Φ' depends on the concrete solutions. In case of the same signs, Eq.(71) resembles the second law of thermodynamics and defines an 'arrow of time' for our dynamical system. If this is not true, the theorem is violated in a very specific way. It may be an interesting point for cosmological modeling.

Our system of equations (70) has other interesting properties. The general solution of the first two equations can be written in terms of integrals of the function $G(\xi)$:

$$\psi(\xi) = \psi_0 + \int_{\xi_0}^{\xi} G, \quad H(\xi) = H_0 \exp\left(-\int_{\xi_0}^{\xi} G^2\right). \quad (73)$$

¹⁷The full system (68)-(69) is evidently over-complete and may be called the extended system. It is useful when we look for integrals of motion. Its subsystems can be used for constructing some special solutions.

¹⁸As far as we are interested in the classical theory we usually will look for integrability in the Liouville sense.

Then, inserting these ‘solutions’ into the third and the forth equations and integrating them we can write one integral equation for $G(\xi)$ instead of system (70):

$$G(\xi) \equiv \frac{\eta}{\chi} = \left(\eta_0 - \frac{1}{2} \int_{\xi_0}^{\xi} U_{\psi} H \right) \left(\chi_0 - \int_{\xi_0}^{\xi} U H \right)^{-1}, \quad (74)$$

where $\psi(\xi)$ and $H(\xi)$ are given by Eq.(73). As was discussed in [29] and [34], the standard (regular and non-degenerate) horizon appears when $\chi_0 = \eta_0 = 0$. Then $h(\xi_0) = 0$ while $G(\xi_0) = U_{\psi}(\xi_0)/2U(\xi_0)$ is finite if $U_{\psi} \neq 0$. It follows that G , ψ , H are finite and can be expanded in convergent series around ξ_0 if the potential $U(\xi, \psi)$ is analytic in a neighborhood of (ξ_0, ψ_0) .¹⁹ When $U_{\psi} \equiv 0$ there is the obvious integral of motion $\eta = \eta_0$. As can be seen from the above equations and was proved in [29], there is no horizon for $\eta_0 \neq 0$ but, in the case of $\eta_0 = 0$, we have $G \equiv 0$ and return to pure dilaton gravity with the solution (67) always having horizons.

In simple cases, the integral equation can easily be reduced to a differential one. For example, if $U = u(\xi) v(\psi)$ and $U_{\psi} = 2g U$, the integral equation can be reduced to the second-order differential equation, which is not integrable for arbitrary $u(\xi)$ but is explicitly integrable if $U_{\xi} = g_1 U$. This result is quite natural as in this case there exist two additional integrals, $\eta = g\chi + \eta_0$ and $\rho = g_1\chi + \rho_0$ and therefore the most direct approach is to use the extended differential system. In non-integrable cases or when there is only one additional integral, the integral equation still can be a quite useful analytical tool, which we describe in a separate publication. Here, we only briefly outline a generalization of the approach of Ref.[29] to finding potentials $U(\xi, \psi)$ for which the extended differential system has additional integrals.

Generalizing the approach of [29] and the above remarks about possible integrals of motion let us collect those equations which can generate such integrals:

$$0 = \rho + UH + \eta^2/\chi = \chi' + UH = \eta' + U_{\psi}H/2 = \rho' + U_{\xi}H = 0, \quad (75)$$

$$0 = \psi\eta' + \psi U_{\psi}H/2 = \eta\psi' - \eta^2/\chi = \xi\rho' + \xi U_{\xi}H = \xi'\rho - \rho = 0, \quad (76)$$

where the first equation in (75) is the energy constraint, which we denote E_0 and the next items in this chain of equations are denoted by E_i , $i = 1, \dots, 7$. Now, taking the sum $\sum_0^7 c_i E_i$ with $c_4 = c_5 = c_6 = c_7 = c_0$ we find that the solutions of equations (75) satisfy the identity

$$[c_1\chi + c_2\eta + c_3\rho + c_4(\psi\eta + \xi\rho)]' = -H[(c_1 + c_4)U + c_2U_{\psi}/2 + c_3U_{\xi} + c_4(\psi U_{\psi}/2 + \xi U_{\xi})]. \quad (77)$$

Therefore, if the r.h.s. identically vanishes, the l.h.s generates the integral of motion,

$$c_1\chi + c_2\eta + c_3\rho + c_4(\psi\eta + \xi\rho) = I_1. \quad (78)$$

This means that for the potentials $U(\xi, \psi)$, satisfying the partial differential equation

$$(c_1 + c_4)U + c_2U_{\psi}/2 + c_3U_{\xi} + c_4(\psi U_{\psi}/2 + \xi U_{\xi}) = 0, \quad (79)$$

there exist the corresponding integral of equations (75). All the above integrals can be obtained by applying this theorem.²⁰ The solution of Eq.(79) depends on an arbitrary function of one variable. Using this fact it is possible, in some simple cases, to derive one more integral.

In the above example of the potential $U = U_0 \exp(2g\psi + g_1\xi)$ with two additional integrals we immediately find the equation for χ ,

$$\chi' = (g^2 + g_1)\chi + (\rho_0 + 2\eta_0 g) + \eta_0^2/\chi, \quad (80)$$

¹⁹This is shown in [34]. In [22] one can find a detailed discussion of regular solution with horizons, including a generalization of the Szekeres - Kruskal coordinates as well as examples of singular and degenerate horizons.

²⁰An interesting exception is the integral (26) in Ref.[29] which apparently requires a more general method for its explanation. E.A.Davydov formulated a group theoretical approach for deriving integrals of motion in general DGS theories, which apparently allows to obtain more general integrals than those discussed here.

by solving of which we explicitly express ξ , η , ρ , h and ψ as functions of χ . This is sufficient for finding the portrait of this physically interesting system, which will be presented elsewhere.

To demonstrate the problems, which remain even for apparently simpler systems, we consider the potential $U(\psi)$ also having two additional integrals. The obvious linear integral is $\rho = \rho_0$. To obtain one more integral suppose that $\psi U_\psi = 2gU$. Then we have the additional bilinear integral of the three differential equations:

$$\psi\eta - (g+1)\chi + \rho_0\xi = I_0, \quad \chi' = \rho_0 + \eta^2/\chi, \quad \eta\psi' = \eta^2/\chi, \quad \psi\eta' = g\chi'. \quad (81)$$

We can exclude ψ (or, η) and thus get two equation for χ and η (or, ψ). However, this is not an integrable dynamical system for two functions because of its explicit dependence on ξ . If $\rho_0 = 0$ in the expression for the integral I_0 , it can be explicitly integrated, like the previous case, but this gives only a ‘partial’ solution. This is a typical problem – bilinear integrals having ξ -depending terms that describe a sort of a ‘back-reaction’ of gravity on matter.

A detailed comparative analysis of most interesting integrable and partially integrable DGS systems will be given in a separate publication. In conclusion of this Section we summarize its main points. Dilaton gravity with scalars is in general not integrable even with formally sufficient number of integrals of motion. The models with massless scalars qualitatively differ from the scalaron models (DSG) that inevitably include non-integrability. Fortunately, in some physically important cases the non-integrable systems are partially integrable and therefore can be effectively studied, at least qualitatively. The solutions near horizons and singularities can be studied analytically, using exact expansions as well as iterations of the master integral equation. On the other hand, our approach to constructing systems with additional integrals may help to find integrable or partially integrable systems qualitatively close to the realistic ones.

6 Summary and outlook

The main new results of this paper are the following. In Section 2, the standard spherical reduction of the D -dimensional DVG and its Kaluza-like cylindrical reduction in the dimensions 3 and 4 are briefly summarized. In Section 3 we derive the transformation of the two-dimensional DVG into the equivalent DSG. This opens a way for applying to the vecton theory some methods developed in two-dimensional dilaton gravity coupled to scalar fields. In particular, the nonlinear kinetic terms of the vecton theory transform into completely standard potentials depending only on scalar fields (dilaton, scalaron, other scalars). The scalar formulation makes it easier to look for additional integrals of motion in the one-dimensional reductions of DSG.²¹ In Section 4 we describe a more standard transition from the two-dimensional DVG to unified description of cosmological and static solutions in arbitrary gauges and write parametrization invariant Hamiltonian equations of one-dimensional DSG theory. The canonical constraint formalism is most convenient in searches for new integrals of motion, which is illustrated by the three-dimensional case. This hints at possible further generalizations of the approach proposed in [29] (pp. 1698-1699), where we derived two nontrivial one-dimensional DGS models having two additional integrals.

Section 5 is a central part of this paper. There we introduce the simplest form of the ordinary differential equations describing the generic DGS theory and derive the most important properties of their cosmological and static solutions. In general, this theory is not integrable and we propose several effective approaches to its analytic and qualitative investigation. The simplest one is the power series expansion near the horizons and generalized expansions near the singularities (taking account of nonanalytic terms). We propose a more powerful approach based on an integral equation

²¹Pure dilaton gravity is a topological theory and thus reduces to the one-dimensional integrable system. There exist nontrivial integrable DGS models involving one massless scalar, [29]. More complex models with effectively massive scalar field may have one additional integral, at best, and thus generally remain non-integrable.

for the most important function of the system. This equation, which we call the *master integral equation*, summarizes important properties of our *cosmo-stat* system. In completely integrable cases it may be reduced to differential dynamical system. In a few instances, it was possible to, find a global analytic solution of the dynamical system and to draw a picture resembling the classical phase portraits. An example of such a portrait is presented in Appendix.

In summary, we stress once more that a main goal of this paper is to find approaches to understanding the global picture of cosmological and static solutions of dilaton gravity that couples to nonlinear scalar fields. These theories are not integrable but, possibly, can be approximated by integrable ones. We have shown that there may exist additional integrals and have proposed some tools for exploiting this fact. Although main features of the vecton theory hint at its relevance to dark energy, inflation,²² and dark matter, the precise relation is still to be uncovered, and the best way to solving this problem is in looking for global qualitative portraits. The difficulty is that the exact system is non-integrable and thus the portrait must be at least three-dimensional.

Understanding global properties of classical solutions is also desirable for their quantization. The simplest approach was attempted some time ago for classically integrable gravitational systems with minimum number of degrees of freedom (see, e.g. [44], [45], [38] and references therein). This primitive quantization can justifiably be criticized for not taking into account space inhomogeneities, which are crucial in cosmological applications (for a review of more sophisticated ideas in quantum gravity see, e.g., [46], [47]). Nevertheless, we hope that such a quantization of DSG might be of some interest in considering properties of simplest quantum cosmological models.

7 Appendix

7.1 Topological portrait

Some linear integrals of Section 5 were found in Ref.[29] for DGS with ψ -independent potentials which, according to (70), have the integral $Z(\varphi) \dot{\psi} = \eta_0$. If also $U_\xi = gU$, we find from (77), (78) one more integral (recall (68):

$$Z\dot{F} - g\dot{\varphi} \equiv \rho - g\chi = C_1. \quad (82)$$

Then it follows that the solution can be expressed in terms of quadratures, see Eqs.(24)-(25) in [29]. An interesting thing not mentioned in [29] is that this integral also exists for ψ -dependent ‘multiplicative’ potentials $U = u(\varphi)v(\psi)$ when $Z(\varphi) \dot{\psi}$ is not constant.

Consider the second integrable model of [29], in which the potential is independent of ψ and depends on an arbitrary function $w(\varphi)$. Then, in Weyl’s frame V and Z are given by

$$U(\xi) \equiv ZV = [g_1 w(\xi) + g_2 w^{-1}(\xi)]', \quad [2g_3 l(\xi) + g_4 l^2(\xi)] = \xi, \quad (83)$$

where $d\xi \equiv d\varphi/Z(\varphi)$, $l(\xi) \equiv \ln w(\xi)$, and evidently,

$$l(\xi) = -(g_3/g_4) \left[1 \mp \sqrt{1 + \xi (g_4/g_3^2)} \right] \rightarrow \xi/2g_3, \quad \text{if } g_4 \rightarrow 0. \quad (84)$$

The third integral was derived by a ‘brute force’ in [29]. In notation of the present paper it is

$$[\rho/l'(\xi)]^2 - 4g_2 h + 2g_4 \eta_0^2 \ln h = C_1. \quad (85)$$

The full Lagrangian and the meaning of the coupling constants can be found in the quoted reference. As shown there, one can derive the solution in quadratures, which are not elementary if $g_4 \neq 0$. If we take $g_4 = g_2 = 0$, we can obtain a simple solution expressing w as a function of h .

²²While the relation to dark energy models is evident, the inflation was clearly demonstrated only in numerical studies of the massive vector meson theory in the tachyonic case $m^2 < 0$, [43].

Introducing the scales w_0 and h_0 for $w(\xi)$ and $h(\xi)$ we find the relation between normalized w and h , which depends only on one parameter δ defined by the relation:

$$2\delta + 1 = \sqrt{1 - 8g_3 \eta_0^2 / C_1} \geq 0, \quad (86)$$

where $g_3 < 0$ for normal scalars and $C_1 > 0$. Then the (h, w) -portrait of our system is given by

$$w = \frac{|h|^\delta}{|1 + \epsilon|h|^{1+2\delta}|}, \quad (87)$$

where $\epsilon \equiv h/|h|$, $-1 < h < \infty$, $0 < w < \infty$. Now it is not very difficult to draw the picture of the curves describing all possible solutions. In the domain $h < 0$ we have static solutions, while for cosmological ones $h > 0$. The picture looks like a phase portrait of a dynamical system in the (h, w) -plane, with singular points: $(0, 1)$, $(1, \frac{1}{2})$, $(0, 0)$, $(0, \infty)$, $(-1, \infty)$, $(\infty, 0)$. These points are joined by the important separating curves. The most interesting points are: the node of the initial singularity, $(0, 0)$, the saddle point of the horizon, $(0, 1)$, and the most interesting ‘cosmological’ point, $(1, \frac{1}{2})$, at which all cosmologies tangentially coincide.²³

This *topological portrait* describes qualitative properties of static and cosmological solutions for different values of the parameter δ , which characterizes the energy of the massless scalar field. If we deformed it by applying any continuous differentiable transformation preserving the singularities, it will represent essentially the same cosmo-static system. For example, it is easy to move all singularities to a finite domain of the plane. Moreover, to better understand the topological structure of the space of the solutions one may try to extend the portrait to the domains where, say, $2\delta + 1 < 0$ or $h < -1$, etc. We believe that the topological portrait is a most adequate global representation of integrable gravitational systems studied in this paper.

7.2 Dimensions and Lagrangians

The dimensions of geometrical and physical fields and parameters were discussed in [19]. However, in different papers on the affine models we used somewhat different conventions and notation. Here we first give several comments on these matters. In the pure geometrical part, there is no problem at all as we have only the dimension of length. Thus $[s_{ij}] = [a_{ij}] = L^{-2}$, $[\bar{l}] = L^0$, geometric Lagrangian (1) is of dimension L^{-D} , and geometric action is therefore dimensionless. In transition to a physical picture, we define dimensionless tensor densities \mathbf{g}^{ij} , \mathbf{f}^{ij} by varying Lagrangian (1). To make the densities dimensionless we multiply it by a constant $2\tilde{\gamma}$ where $[\tilde{\gamma}] = L^{D-2}$:

$$\mathcal{L} \equiv 2\tilde{\gamma}\sqrt{-\Delta_s}; \quad \frac{\partial \mathcal{L}}{\partial s_{ij}} \equiv \mathbf{g}^{ij}, \quad \frac{\partial \mathcal{L}}{\partial a_{ij}} \equiv \mathbf{f}^{ij}. \quad (88)$$

Using this definition it is not difficult to prove the following identity (l is a dimensionless constant):

$$|\Delta_g| \equiv |\det(\mathbf{g}^{ij} + l \mathbf{f}^{ij})| = \tilde{\gamma}^D |\det(s_{ij} + l^{-1} a_{ij})|^{(D-2)/2}. \quad (89)$$

With this identity, one can find the conjugate Lagrangian density $\mathcal{L}^* = \mathcal{L}^*(\mathbf{g}^{ij}, \mathbf{f}^{ij})$:

$$\mathcal{L}^* = \nu^{-1} (-\Delta_g / \tilde{\gamma}^2)^\nu; \quad s_{ij} = \frac{\partial \mathcal{L}^*}{\partial \mathbf{g}^{ij}}, \quad a_{ij} = \frac{\partial \mathcal{L}^*}{\partial \mathbf{f}^{ij}}. \quad (90)$$

This result was apparently known to Einstein in the special (four-dimensional) case, when $\nu = 1/2$ and $l = 1$. Following his approach we find the ‘physical’ Lagrangian (2) as described in [18] - [22].

²³Even more interesting but much more complex is the portrait of the model with massless vector (when $g_2 \neq 0$) and scalar, see Eq.(30) in [29]. We will discuss it in a separate publication.

Eqs.(88) - (90) play the key role in this derivation. When considering the physical Lagrangian it is better to use the gravitational constant $\kappa \equiv G/c^4$ to define the physical fields A_i and F_{ij} proportional to a_i and f_{ij} but having the standard dimensions and satisfying the relation $[\kappa A_i^2] = 1$, $[\kappa F_{ij}^2] = L^{-2}$, [19]. In this paper we choose the units $c = \kappa = 1$ and thus A_i is dimensionless and F_{ij} is of dimension L^{-1} . In fact, we use notation a_i for the dimensionless potential in this system of units and f_{ij} for the corresponding field tensor $f_{ij} = a_{i,j} - a_{j,i}$; this means that $[\lambda] = L$ and $[\lambda^2 \Lambda] = 1$. In Eq.(2) and in the body of this paper we always apply this agreement on notation. As we do not use the tensor densities in the main text, the bold-face notation is used only to $f_{ij} f^{ij} \equiv \mathbf{f}^2$ and $a_i a^i \equiv \mathbf{a}^2$.

One final remark on the dimensions of constants and fields. In our units, we have essentially three dimensional constants of geometric origin: $\Lambda, \lambda, \mu \equiv m^2$ (k_ν and $\tilde{\gamma}$ are auxiliary dimensional constants having no relation to affine geometry). Coordinates t, r, u, v are of dimension L , the main fields a_i, ψ, φ are dimensionless while $[q] = L^{-1}$. When we denote the scalaron by ψ (in Section 5) we silently make it dimensionless by including m in its definition, $\psi = q/m$.

7.3 Three-dimensional pure vecton theory

Let us consider the vecton Lagrangian (42) in the LC gauge (when $\alpha = \gamma$ and $2\alpha = \ln|h| = F$) and without transition to the scalaron description

$$\mathcal{L}_v^{(1)} = \lambda_0 \varphi \dot{a}_\epsilon^2 / h - \dot{\varphi} \dot{F} - \mu \varphi a_\epsilon^2 - 2\Lambda \varphi h, \quad (91)$$

where $h = \epsilon e^{-F}$ and $\lambda_0 \equiv \lambda^2 \Lambda$ are dimensionless while $\mu = m^2$ is of dimension L^{-2} . The Hamiltonian constraint that must be derived before passing to LC coordinates is

$$\mathcal{H}_v^{(1)} = \lambda_0 \varphi \dot{a}_\epsilon^2 / h - \dot{\varphi} \dot{F} + \mu \varphi a_\epsilon^2 + 2\Lambda \varphi h = 0 \quad (92)$$

while other equations are simply derived by varying Lagrangian (91). Instead of the Hamiltonian equations we write the equations of motion in the first order form similar to (70). Defining $\xi \equiv (\varphi^2/2)$, $(\varphi \dot{a})/h \equiv b$, $A \equiv a/\chi$, $\mu_1 \equiv \mu/\lambda_0$, and denoting $d/d\xi$ by the prime we find:

$$b' = -\mu_1 A, \quad H' = \mu A^2 H, \quad 2\xi a' = bH, \quad \xi \chi' = (2\Lambda \xi + \lambda_0 b^2/2) H, \quad (93)$$

where we omitted index ϵ of a and A . In the massless limit $\mu = \mu_1 = 0$ and therefore $H = H_0$, $b = b_0$ are independent of ξ . This immediately gives the three-dimensional Maxwell - Einstein solution. Indeed, we see that

$$h(\xi) = H_0 \chi(\xi), \quad a = a_0 + (b_0 H_0/2) \ln \xi/\xi_0, \quad \chi = \chi_0 + (\lambda_0 b_0^2 H_0/2) \ln \xi/\xi_0 + 2\Lambda(\xi - \xi_0), \quad (94)$$

and thus returning to the variable φ we find (taking $\chi_0 = 0$ and denoting $\lambda_0 b_0^2 H_0 \equiv \tilde{c}$):

$$h = H_0 \int_{\xi_0}^{\xi} d\xi \left(\frac{\tilde{c}}{2\xi} + 2\Lambda \right) = H_0 \int_{\varphi_0}^{\varphi} d\varphi \left(\frac{\tilde{c}}{\varphi} + 2\Lambda \varphi \right). \quad (95)$$

Equations (94)-(95) define the general solution of the pure dilaton gravity with the potential proportional to the expression in the brackets, see Eq.(67).

To find approximate solutions for small but finite values of μ, μ_1 we may take $H = H_0, b = b_0$, expressions (94) for $a_0(\xi), \chi_0(\xi), A_0 = a_0/\chi_0$ and thus construct successive approximations for $b_i, H_i, a_i, \chi_i, A_i = a_i/\chi_i$. Our equations (93) or their integral formulation are convenient for deriving the solution near horizons and near singularities at $\xi \rightarrow 0, \infty$. In fact, they are simpler than the approximate equation studied in [18] - [21]. At the horizon at $\xi = \xi_0$, when $a_0 = \chi_0 = 0$ and H_0, A_0 are finite, we have the convergent expansions of the solution in powers of $\xi - \xi_0$. Moreover, one can see that the radius of convergence cannot be larger than ξ_0 (it can be lesser

if there is a second horizon). Near singularities we can also find expansions in powers of ξ if we correctly take into account the logarithmic terms seen in the zeroth approximation (95).

A better though much more complex iterations can be derived with the aid of the integral equations for $A(\xi)$ similar to (73), (74), if we take the zeroth approximation $A = A_0$, solve the equations for H and b , solve the resulting equation for $A(\xi)$, and then repeat this cycle. A simpler approach is to first find and use an additional approximate integral. On this way, the scalaron formulation looks more promising. For example, we have seen that the potential for $D = 3$ depends on q^2/ξ and there exists the additional integral (64), which presumably can be used for constructing more effective approximations.

Acknowledgment

This work was supported in part by the Russian Foundation for Basic Research: Grant No. 11-02-01335-a and Grant No. 11-02-12232-ofi-M-2011. Useful remarks of E.A. Davydov are kindly acknowledged.

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